HOW TO CLASSIFY GEOMETRIC INVARIANTS

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Abstract. Given a particular kind of geometry (Riemannian, Complex, foliation, ...), it is of fundamental importance to determine when two objects are isomorphic. The method of equivalence is a systematic way to classify the local invariants of a particular geometry. The general approach is to rewrite our geometric objects as G-structures, where G is a Lie group that captures the essence of the geometry. We may then compute something called Spencer cohomology of G, which tells us what the local invariants are. After outlining the general theory I will cover several examples. Among these we will see why curvature is the only local invariant in Riemannian geometry.

1. G-Structures

One of the fundamental problems in geometry is that of equivalence, the problem of determining when two objects in a geometric category are isomorphic. While there is no general definition of what a 'geometric' structure on manifolds is, there are common features which all of the classical geometric structures have. One of the way to capture what is meant by a 'geometric' structure is a G-structure, where G is a Lie group. The choice of G determines a kind of geometry, in the sense that G is the group of 'local symmetries' of your geometry, or the group which preserves the framings compatible with a geometry. To define what a G-structure is we need first a definition.

Definition. Given a smooth manifold M^n, a *coframe* at $x \in M$ is an isomorphism $u\colon T_xM \to \mathbb{R}^n$. We denote the set of coframes based at x by $\mathcal{F}_x^{\mathrm{GL}}$. The *frame bundle* $\mathcal{F}^{\mathrm{GL}}$ is then given by the union $\cup_{x\in M}\mathcal{F}_x^{\mathrm{GL}} \subset T^*M \otimes \mathbb{R}^n$. The set $\mathcal{F}^{\mathrm{GL}}$ is a sub-bundle of $T^*M \otimes \mathbb{R}^n$ whose projection map we denote by π.

It is not difficult to see that this is a principal $\mathrm{GL}(\mathbb{R}^n)$ bundle. Indeed, for frames $u, v \in \mathcal{F}_x^{\mathrm{GL}}$ the map $A = vu^{-1} \in \mathrm{GL}(\mathbb{R}^n)$ is the unique element of $\mathrm{GL}(\mathbb{R}^n)$ so that $Au = v$.

Definition. Let $G \subset \mathrm{GL}(\mathbb{R}^n)$ be a Lie subgroup. A *G-structure* on M^n is a principal G subbundle of $\mathcal{F}^{\mathrm{GL}}$.

G-structures arise whenever we have a geometric structure on M which is preserved by G. To illustrate, consider Riemannian geometry, which corresponds to $O(n)$-structures.

Example. A Riemannian metric on a manifold M is a choice of a non-degenerate, symmetric bi-linear form g_x at each point of M. We give $\mathbb{R}^n$ the standard Euclidean structure. Then we consider the subset $\mathcal{F}^{O(n)}$ of coframes $u \in \mathcal{F}^{\mathrm{GL}}$ so that $u\colon T_xM \to \mathbb{R}^n$ is an isometry. Given two frames $u, v \in \mathcal{F}^{O(n)}$, the element vu^{-1} preserves the inner product on $\mathbb{R}^n$, so must be an element of $O(n)$. Thus $\mathcal{F}^{O(n)}$ is an $O(n)$ structure on M.

Conversely, given an $O(n)$-structure $\mathcal{F}$ on M we can determine a unique metric on M as follows. At each point x in M choose any u in the fiber $\mathcal{F}_x$. We let g_x be the pullback of the fixed inner product on $\mathbb{R}^n$ along u. Because any other coframe differs from u by an orthogonal element, this is independent of our choice of coframing u. Smoothness of g_x follows from smoothness of $\mathcal{F}$.

A frequently convenient way to describe a G-structure $E \to M$ is to give a section $\sigma\colon M \to E$. The fiber over a point $x \in M$ is then the orbit of $\sigma(x)$ under the action of G. For example, in Riemannian geometry one often gives an orthonormal coframing of the manifold instead of explicitly writing out the metric.

2. Equivalence

Any diffeomorphism $f\colon M^n \to N^n$ has a canonical lift $f^1\colon \mathcal{F}^{\mathrm{GL}}(M) \to \mathcal{F}^{\mathrm{GL}}(N)$ given by $f^1(u) = u \circ (f^{-1})'$. This map is a bundle isomorphism, which allows us to describe when two G-structures are equivalent.

Definition. Let $E \to M$ and $F \to N$ be G-structures. A diffeomorphism $f\colon M \to N$ is an *equivalence* of G-structures if $f^1(E) = F$.

As mentioned before, we would like a method to determine when two G-structures are equivalent. For simplicity (but ultimately without any real loss of generality.) let us consider the question of when a G-structure is equivalent to the flat model, defined as

Definition. A G-structure $E \to M$ is *flat* if each point lies in a coordinate chart $(x^1, \ldots, x^n)$ so that the coframing $(dx^1, \ldots, dx^n)_p \in E_p$ for each point p in the coordinate domain.

To show that a structure is flat we need to find a map into $\mathbb{R}^n$ which satisfies conditions on its derivatives. In other words, a given G-structure is flat exactly when a certain overdetermined PDE has a solution. This is where the theory of Exterior Differential Systems excels, so it makes sense to use machinery from the general theory of EDS. In particular, the Spencer cohomology will tell us what the obstructions to flatness are.

3. Tableaux

The PDE involved in determining flatness are first order linear equations. In the real analytic category solutions to such equactions are determined through the study of tableaux.

Suppose as given vector spaces V and W with respective bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_s$ and coordinates $x = x^i v_i$, $u = u^a w_a$. Given a system of constant coefficient, first order, homogeneous PDE on functions f from W to V,

$$B^\lambda(f) = B^{\lambda i}_a \frac{\partial f^a}{\partial x^i} = 0,$$

we define a linear subspace B of $W^* \otimes V$ spanned by elements $B^{\lambda i}_a w^a \otimes v_i$ (a raised subscript denotes the dual basis.). B is the space of equations for our PDE, and the dual space $B^\perp \subset W \otimes V^*$ is the space of linear solutions, called a *tableau*.

Each of these steps can be reversed, so a tableau is the same data (in a dual form) as a constant coefficient, first order, homogeneous PDE. The reason for considering tableaux is that the existence of analytic solutions becomes a problem in linear algebra. Indeed, if the series

$$f(x) = (p^a + p^a_i x^i + p^a_{ij} x^i x^j + \ldots) w_a$$

is a solution then each homogeneous term must be as well. The identification of $\mathbb{R}[x_1, \ldots, x_n]$ with $S(V^*) = \oplus_{k=0}^\infty \operatorname{Sym}^k V^*$ suggests the following

Definition. Given a tableau $A \subset W \otimes V^*$, the q-th prolongation is

$$A^{(q)} = (A \otimes (V^*)^{\otimes q}) \cap (W \otimes \operatorname{Sym}^{q+1} V^*).$$

An alternative, equivalent definition is to let $A^{(0)} = A$ and [1]

$$A^{(q+1)} = \left\{ P \in W \otimes \operatorname{Sym}^{q+1} V^* \colon \frac{\partial}{\partial x^i} P \in A^{(q)} \text{ for all } i \right\}$$

From this definition one sees that $A^{(q)}$ is the space of homogeneous degree q solutions to the associated PDE.

Now let us return to the case of a possibly flat G-structure. By definition the group G is a matrix Lie group with a representation into $\mathrm{GL}(V)$, $V = \mathbb{R}^n$. Consequently, the Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{gl}(V) = V \otimes V^*$, which we consider as a tableau. The prolongations of $\mathfrak{g}$ are intimately tied to the invariants of G-structures via the Spencer cohomology of $\mathfrak{g}$, defined below. First an example which we will use later.

Example. In the case of Riemannian geometry $\mathfrak{g} = \mathfrak{so}(n) \subset V \otimes V^*$. Because we have a metric we may lift indices to identify $V \otimes V^*$ with $V^* \otimes V^*$. Under this identification $\mathfrak{so}(n)$ is mapped to $\Lambda^2 V^*$. This implies that $\mathfrak{so}(n)^{(1)} = (\Lambda^2 V^* \otimes V^*) \cap (V^* \otimes \operatorname{Sym}^2 V^*) = 0$. Indeed, an element $A_{ijk} v^i \otimes v^j \otimes v^k \in \mathfrak{so}(n)^{(1)}$ is anti-symmetric in the first two indices and symmetric in the last two. Thus

$$A_{ijk} = -A_{jik} = -A_{jki} = A_{kji} = A_{kij} = -A_{ikj} = -A_{ijk}.$$

From the second definition of prolongation we see immediately that $A^{(q)} = 0$ for $q > 0$.

4. Spencer Cohomology

Although it would take at least another talk to explain the specifics, the problem of determining whether a G-structure is flat comes down to the existence of a solution to a particular system of PDE. In the general theory of Exterior Differential Systems there is a very powerful tool, Spencer Cohomology, for determining all of the obstructions to solutions.

To define Spencer Cohomology, we define the free Spencer Complex as the complex $C^{p,q} = \operatorname{Sym}^p V^* \otimes \Lambda^q V^*$ for $p, q \geq 0$ and differential on $P \otimes dx^{i_1} \wedge \ldots \wedge dx^{i_q} \in C^{p.q}$ given by

$$d(P \otimes dx^{i_1} \wedge \ldots \wedge dx^{i_q}) = \sum_{i=1}^{n} \frac{\partial P}{\partial x^i} \otimes dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_q}$$

Under the identification of $\operatorname{Sym}^*(V^*)$ with $\mathbb{R}[x_1, \ldots, x_n]$ this is exactly the exterior derivative of polynomial forms.

Theorem (Polynomial Poincare lemma)**.** *The homology $H^{p,q}(C^{*,*}) = 0$ for $p+q > 0$ and $H^{0,0}(C^{*,*}) = \mathbb{R}$.*

Proof. Consider the vector field $\xi = x^i \partial_{x^i}$ on $\mathbb{R}^n$. By Cartan's formula for a form $\omega \in C^{p,q}$ we have

$$d \circ i_\xi \omega + i_\xi \circ d\omega = \mathcal{L}_\xi \omega = (p+q)\omega.$$

In other words, for $p + q > 0$ the identity map is chain homotopic to zero. □

Now, we may tensor $C^{p,q}$ with the vector space W and consider the elements of the W factor as constants. The Polynomial Poincare lemma still holds with the difference that $H^{0,0}(W \otimes C^{*,*}) = W$.

Finally, since $C^{p+1,q}(A) := A^{(p)} \otimes \Lambda^q V^* \subset W \otimes C^{p+1,q}$ we may consider this as a subcomplex of $W \otimes C^{*,*}$. It is straightforward from the second definition of prolongation

[1] Under the identification of $\mathbb{R}[x_1, \ldots, x_n]$ with $S(V^*)$ we can define partial derivatives $\frac{\partial}{\partial x^i} \colon \operatorname{Sym}^{q+1} V^* \to \operatorname{Sym}^q V^*$.

that the differential is well defined on the restriction. The cohomology of this complex is where obstructions to flatness live.

Considering $A = \mathfrak{g}$, we get a diagram such as:

$$\begin{array}{ccccccccc}
\mathfrak{g}^{(2)} & & \mathfrak{g}^{(2)} \otimes V^* & & \mathfrak{g}^{(2)} \otimes \Lambda^2 V^* & & \mathfrak{g}^{(2)} \otimes \Lambda^3 V^* & & \cdots \\
& \searrow & & \searrow & & \searrow & & \searrow & \\
\mathfrak{g}^{(1)} & & \mathfrak{g}^{(1)} \otimes V^* & & \mathfrak{g}^{(1)} \otimes \Lambda^2 V^* & & \mathfrak{g}^{(1)} \otimes \Lambda^3 V^* & & \cdots \\
& \searrow & & \searrow & & \searrow & & \searrow & \\
\mathfrak{g} & & \mathfrak{g} \otimes V^* & & \mathfrak{g} \otimes \Lambda^2 V^* & & \mathfrak{g} \otimes \Lambda^3 V^* & & \cdots \\
& \searrow & & \searrow & & \searrow & & \searrow & \\
V & & V \otimes V^* & & V \otimes \Lambda^2 V^* & & V \otimes \Lambda^3 V^* & & \cdots
\end{array}$$

We can now roughly state the following theorem.

Theorem. *For fixed G, all the obstructions to a G-structure being flat are contained in $H^{p,2}(\mathfrak{g})$ for $p \geq 0$.*

What this means is, given a particular G-structure $E \to M$, at each point $x \in M$ there is an element (curvature) of $H^{p,2}(\mathfrak{g})$ for each $p \geq 0$ and E is flat at x exactly if each of these elements is zero in a neighborhood of x. Of course, for most Lie groups the groups $H^{p,2}(\mathfrak{g})$ will vanish for large enough p so there will be finitely many curvature conditions to check.

Example. In the case of Riemannian geometry $\mathfrak{g} = \mathfrak{so}(n)$. We have already seen that $\mathfrak{g}^{(p)} = 0$ for $p > 0$, so we need only compute $H^{p,2}$ for $p = 0, 1$.

The map $\Lambda^2 V^* \otimes V^* \to V^* \otimes \Lambda^2 V^*$ is an isomorphism (check?), so $H^{0,2} = 0$. This corresponds to the existence and uniqueness of the Levi-Civita connection.

The group $H^{1,2}$ is the kernel of the map $\Lambda^2 V^* \otimes \Lambda^2 V^* \to V^* \otimes \Lambda^3 V^*$, the Bianchi identity.

In the case that we want to determine if two non-flat G-structures M and N (of dimension n) are equivalent we look at the product $M \times N$ and consider the sub-manifold where the differences of the curvature elements is zero. If there is an n-dimensional sub-manifold which is transverse to both projections then locally it will describe a graph of a diffeomorphism between M and N. This diffeomorphism will be a G-isomorphism from part of M to part of N.